Linear Functional

A continuous linear transformation defined from a NLS N unto $\mathbb{R}$ or $\mathbb{C}$ is called a continuous linear functional or a functional

i.e. if T is a L.T. s.t.

$$T : N \to \mathbb{R} \text{ (or } \mathbb{C})$$

Then T is a functional

Here $\mathbb{R}$ is set of real numbers

($\mathbb{C} \to$ set of complex numbers)

Note→ $\mathbb{R}$ & $\mathbb{C}$ are NLS

Note→ Set of all cont. L.T. → N^* → conjugate space (or Adjoint space)

Hahn Banach Theorem (statement)

Let M be a linear subspace of a NLS N and let f be a functional defined on M. Then f can be extended to a functional F defined on the whole space N such that $\|F\| = \|f\|$

Th1→ Let N be a NLS and x_0 be a non-zero vector in N, then $\exists$ a functional F in N^* such that —

$$F(x_0) = \|x_0\| \text{ and } \|F\| = 1$$

In particular, if $x \neq y$ $(x, y \in N)$ then $\exists$ an $f \in N^*$ s.t. $f(x) \neq f(y)$

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Pf → Let $M = \{\alpha x_0\}$ be a linear subspace of N. (28)
It is spanned by x_0 as $x_0 \neq 0$.
Define f_0 on M s.t. $f_0(\alpha x_0) = \alpha \|x_0\|$ →(1)
we shall show that f_0 is a functional on M s.t. $\|f_0\| = 1$. Then we shall extend f_0 by Hahn Banach theorem.
To prove that f_0 is functional, we need to prove f_0 to be linear, bdd.
Let $y_1, y_2 \in M$. Then $y_1 = \alpha x_0$ & $y_2 = \beta x_0$ for some scalars α & β.
Now if γ, δ are any scalars, then —

$$f_0(\gamma y_1 + \delta y_2) = f_0(\gamma \alpha x_0 + \delta \beta x_0)$$
$$= f_0((\gamma\alpha + \delta\beta) x_0)$$
$$= (\gamma\alpha + \delta\beta)\|x_0\| \quad \text{, from (1)}$$
$$= \gamma(\alpha\|x_0\|) + \delta(\beta\|x_0\|)$$
$$= \gamma f_0(\alpha x_0) + \delta f_0(\beta x_0)$$
$$= \gamma f_0(y_1) + \delta f_0(y_2)$$

∴ f_0 is linear.
Now to show f_0 is bdd.
Let $y = \alpha x_0 \in M$. Then $\|y\| = \|\alpha x_0\|$
$= |\alpha| \|x_0\|$.
Now →
$$|f_0(y)| = |f_0(\alpha x_0)| = |\alpha \|x_0\|| = |\alpha| \|x_0\|$$
$$= \|y\| < |y|$$ →(2)
⇒ f_0 is bdd.

Further, $\|f_0\| = \sup\{|f_0(y)| : y \in M, \|y\| = 1\}$

$= \sup\{\|y\| : y \in M, \|y\| \leq 1\}$ — from (2)

Also, $= 1$

$f_0(x_0) = \|x_0\|$, by def^n of 'f_0' if we take $\alpha = 1$

$\therefore$ By Hahn-Banach thm, f_0 can be extended to a norm preserving functional $F \in N^*$ s.t.

$F(x_0) = f_0(x_0) = \|x_0\|$

and $\|F\| = \|f_0\| = 1$

In particular case, since $x \neq y$,

$\therefore x - y \neq 0$ and so by above argument

$\exists$ an $f \in N^*$ s.t.

$f(x-y) = \|x-y\| \neq 0 \Rightarrow f(x) - f(y) \neq 0$

$\Rightarrow f(x) \neq f(y)$

(Proved)

<u>Thm</u> Let M be a closed linear subspace of a NLS N and x_0 a vector s.t. $x_0 \notin M$. Then there exists a functional F in N^* s.t.

$F(M) = \{0\}$ and $F(x_0) \neq 0$

<u>Proof</u> → Consider the natural mapping →

$\phi : N \to N/M$

s.t. $\phi(x) = x + M$

Then ϕ is a cont. linear transformation.

If $m \in M$ then $\phi(m) = m + M = 0 \rightarrow$ Zero vector M in N/M.

i.e. $\phi(M) = \{0\}$ $\rightarrow (1)$

Now, since $x_0 \notin M$, $\therefore$

$\phi(x_0) = x_0 + M \neq 0$ ($\neq$ zero vector M)

$\therefore$ $\exists$ a functional $f \in (N/M)^*$

s.t. $f(x_0 + M) = \|x_0 + M\| \neq 0$ $\rightarrow (2)$

We now defined F by:

$$F(x) = f(\phi(x))$$

Then F is a linear functional on N

F is linear

$$F(\alpha x + \beta y) = f(\phi(\alpha x + \beta y))$$

$$= f((\alpha x + \beta y) + M), \quad \because \phi(x_0) = x_0 + M$$

$$= f(\alpha(x + M) + \beta(y + M))$$

$$= \alpha f(x + M) + \beta f(y + M), \because$$

$$= \alpha f(\phi(x)) + \beta f(\phi(y))$$

$$= \alpha F(x) + \beta F(y)$$

F is bdd.

$$|F(x)| = |f(\phi(x))| \leq \|f\| \, \|\phi(x)\|$$

$$\leq \|f\| \, \|\phi\| \, \|x\|$$

$$\leq \|f\| \, \|x\|, \quad \because \|\phi\| \leq 1$$

Since f is bdd

$\therefore$ F is also bdd. $\Rightarrow F$ is a functional on N i.e. $F \in N^*$

Further, $m \in M$ then $F(m) = f(\phi(m)) = f(0) = 0$ & $F(x_0) = f(\phi(x_0)) = f(x_0 + M) \neq 0$, from (2)

(Proved)